TOPOLOGICAL ANOMALIES meets GRAPH COMPLEX HOMOLOGY

presented on A POSTER FULL OF FORMULAS based on original research by PAUL-HERMANN BALDUF and **SIMONE HU** at the Mathematical Institute of the UNIVERSITY OF OXFORD and reported in ARXIV 2503.09558.

Propagator in topological QFT

Position space

- n-dimensional topological QFT, position space $\vec{x} = (x^{(1)}, \ldots, x^{(n)})^\intercal$, with field differential operator = de Rham operator
$$\mathrm{d} = \mathrm{d}x^{(1)}\partial_{x^{(1)}} + \mathrm{d}x^{(2)}\partial_{x^{(2)}} + \ldots + \mathrm{d}x^{(n)}\partial_{x^{(n)}}.$$
- Propagator is Green function of d, defined by $\mathrm{d}P_n(\vec{x}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,\delta^n(\vec{x})\ \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. It is
$$P_n(\vec{x}) = \frac{\Omega_n}{|\vec{x}|^n} = \frac{\sum_{j=1}^n (-1)^j x^{(j)}\,\mathrm{d}x^{(1)} \wedge \widehat{\mathrm{d}x^{(j)}} \wedge \mathrm{d}x^{(n)}}{\sqrt{\vec{x}\cdot\vec{x}}^{\,n}}.$$
Ω_n is the projective n-dimensional volume form
- Examples: $P_1 = \frac{x}{|x|} = \mathrm{sgn}(x)$,
$$P_2 = \frac{x^{(2)}\,\mathrm{d}x^{(1)} - x^{(1)}\,\mathrm{d}x^{(2)}}{{x^{(1)}}^2 + {x^{(2)}}^2} = \frac{r^2\sin^2\varphi\,\mathrm{d}\varphi + r^2\cos^2\varphi\,\mathrm{d}\varphi}{r^2} = \mathrm{d}\varphi.$$

Parametric space

- Recall integral repr. of Euler gamma function,
$$\frac{1}{|\vec{x}|^n} = \frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty \frac{1}{a^{\frac{n}{2}+1}} e^{-\frac{\vec{x}^2}{a}}\,\mathrm{d}a.$$
- For each component $x^{(j)}$ introduce $s^{(j)} := \frac{x^{(j)}}{\sqrt{a}}$. Then $\mathrm{d}s^{(j)} = \frac{\mathrm{d}x^{(j)}}{a^{\frac{1}{2}}} - \frac{x^{(j)}}{2a^{\frac{3}{2}}}\mathrm{d}a$ [GKW25; Bud+23]. Wedge product:
$$\mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\mathrm{d}x^{(1)} \wedge \ldots \wedge \mathrm{d}x^{(n)}}{a^{\frac{n}{2}}} + \frac{\mathrm{d}a \wedge \Omega_n}{2a^{\frac{n}{2}+1}}.$$
- If one integrates a, the first term vanishes and
$$\int_0^\infty e^{-\vec{s}^2}\,\mathrm{d}s^{(1)} \wedge \ldots \wedge \mathrm{d}s^{(n)} = \frac{\Gamma(\frac{n}{2})}{2}\frac{\Omega_n}{(\vec{x}^2)^{\frac{n}{2}}} = \frac{\Gamma(\frac{n}{2})}{2}P_n(\vec{x}).$$
- Notice that the integrand factorizes: Each of the n directions contributes $e^{{s^{(j)}}^2}\,\mathrm{d}s^{(j)}$.

Parametric integrals for anomalies: The topological form

- BRST formalism: Differential Q , gauge-invariant "physical" observables A are 0th homology group. That is,
$$QA = 0 \qquad \text{and} \qquad \nexists B : A = QB.$$
- A classically gauge invariant observable might violate gauge invariance at quantum level ("anomaly"). Work in perturbation theory, let $\mathcal{O}_j$ be local operators. Define *bracket* [GKW25]
$$\{\mathcal{O}_1, \ldots, \mathcal{O}_k\} := Q\left(\int_{\mathbb{R}^{n(k-1)}} \mathcal{O}_1 \cdots \mathcal{O}_k\right).$$
- The integral is a sum over Feynman integrals with k vertices in the n-dimensional TQFT,
$$\{\mathcal{O}_1, \mathcal{O}_2, \ldots\} := Q\left(\int_{\mathbb{R}^{n(k-1)}} \mathcal{O}_1 \cdots \mathcal{O}_k\right) = \sum_{\text{Graphs } G} \frac{1}{|\mathrm{Aut}(G)|} I_G \prod_{v \in V_G}\prod_i \varphi_{i,v}.$$
symmetry factor — Feynman integral — External leg structure
- Parametric integrand factorizes along dimension $\Rightarrow$ consider 1-dimensional integrand α_G. Schwinger parameter a_e for each edge. Coordinates $x_e^\pm \in \mathbb{R}$. Then $I_G = \int \alpha_G \wedge \alpha_G \wedge \ldots$ with the **topological form**
$$\alpha_G := \frac{1}{\pi^{\frac{|E_G|}{2}}}\int \cdots \int_{\mathbb{R}^{|V_G|-1}} \bigwedge_{e \in E_G} e^{-s_e^2}\,\mathrm{d}s_e \qquad \text{(differential form of degree } \ell\text{)}, \quad \text{where} \quad s_e := \frac{x_e^+ - x_e^-}{\sqrt{a_e}}.$$
- Key results of [BG25]: Topological form is given by graph matrices and Dodgson polynomials
$$\alpha_G = \frac{1}{\pi^{\frac{\ell}{2}} 4^\ell \left(\frac{\ell}{2}\right)!\,\psi_G^{\frac{\ell+1}{2}}} \sum_{T \text{ spanning tree}} \det\left(\mathbb{I}[T]\right)\left(\sum_{\sigma \in \mathfrak{S}_{\overline{T}}} \psi_G^{\sigma(f_1),\sigma(f_2)} \cdots \psi_G^{\sigma(f_{\ell-1}),\sigma(f_\ell)}\right)\bigwedge_{f \notin T}\mathrm{d}a_f,$$
and $\alpha_G \wedge \alpha_G = 0$ for all graphs (Kontsevich Formality theorem).

Kontsevich formality theorem

- Given is a classical field theory: Field variable $\phi(t,x)$, canonical conjugate $\pi(t,x)$. Hamilton function $H(\phi(t,x), \pi(t,x))$. Poisson bracket $\{f,g\} \in C^\infty$. Equations of motion:
$$\partial_t\phi = \{\phi, H\}, \quad \partial_t\pi = \{\pi, H\}, \quad \{\phi, \pi\} = 1.$$
- Deformation quantisation: Find *star product* $\star$
$$\text{s.t. } [f,g]_\star := f \star g - g \star f \overset{!}{=} \hbar\{f,g\} + \mathcal{O}\left(\hbar^2\right).$$
Should be associative $f \star (g \star h) = (f \star g) \star h$.
- Power series ansatz, differential operators $B_j(f,g)$.
$$f \star g = B_0(f,g) + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \ldots,$$
$B_0(f,g) = f \cdot g$ and $B_1(f,g) = \frac{1}{2}\{f,g\}$. Solution in [Kon03]: Graphs Γ embedded in the upper half plane $\{z \in \mathbb{C} | \Im(z) > 0\}$.
- Angle $\varphi(p,q)$ between geodesic $p \longrightarrow q$ and vertical line $p \longrightarrow i\infty$. Each graph is weighted by an integral $W_G = \text{const} \times \int \bigwedge_{e \in E_G} \mathrm{d}\varphi_e$. Star product is
$$\star = \cdot\ + \sum_{n=1}^\infty \hbar^n \sum_G W_G B_G.$$
- To prove associativity, show vanishing of terms at the boundary:
$$c_G := \int_{\mathbb{R}^{2(|V|-1)}} \bigwedge_e \mathrm{d}\varphi_e = \int_{\mathbb{R}^{2(|V|-1)}} \int_{\sigma_G} e^{\vec{s}^2} \bigwedge_e \mathrm{d}s_e^{(1)}\,\mathrm{d}s_e^{(2)}$$
- Solving position integrals yields $c_G = \int_{\sigma_G} \alpha_\Gamma \wedge \alpha_\Gamma$. So $\alpha_G \wedge \alpha_G = 0$ implies associativity of $\star$.
- Easy proof with $\alpha_G = \phi_G$: $\mathrm{Pf}(A)^2 = \det(A)$, so
$$\begin{aligned}\phi_G \wedge \phi_G &\propto \frac{\left(\mathrm{Pf}\left(\mathrm{d}\Lambda\Lambda^{-1}\mathrm{d}\Lambda\right)\right)^2}{\det\Lambda} = \det\left(\Lambda^{-1}\right)\det\left(\mathrm{d}\Lambda\,\Lambda^{-1}\mathrm{d}\Lambda\right)\\ &= \det\left(\Lambda^{-1}\mathrm{d}\Lambda\,\Lambda^{-1}\mathrm{d}\Lambda\right) = \det\left(\left(\Lambda^{-1}\mathrm{d}\Lambda\right)^2\right)\\ &=: \det(M) = \frac{1}{(\ell/2)!}B_n\left(s_1, s_2, \ldots\right),\end{aligned}$$
where B_n are Bell polynomials and s_j are given by canonical forms (only $\beta^{4k+1} \neq 0$ due to cyclicity of trace and symmetry of Λ):
$$\begin{aligned}s_j &= -\frac{(j-1)!}{2}\mathrm{tr}\left(M^j\right) = -\frac{(j-1)!}{2}\mathrm{tr}\left(\left(\Lambda^{-1}\mathrm{d}\Lambda\right)^{2j}\right)\\ &= -\frac{(j-1)!}{2}\beta_G^{2j} = 0 \quad \forall j \quad \Rightarrow \phi_G \wedge \phi_G = 0.\end{aligned}$$

"The topological form is the Pfaffian form"

Let $\mathcal{C}$ be any choice of cycle incidence matrix and $\mathcal{P}$ any choice of path matrix, then $\det\left(\mathcal{C}\,|\,\mathcal{P}\right) \in \{+1, -1\}$ and
$$\text{“Topological form”} \longrightarrow \quad \alpha_G = \frac{\det\left(\mathcal{C}\,|\,\mathcal{P}\right)}{2^\ell}\ \phi_G \quad \longleftarrow \text{“Pfaffian form”}$$

Wait, what is a Pfaffian?

- Let M be a $2n \times 2n$ skew-symmetric matrix with commuting entries. The *Pfaffian* is
$$\mathrm{Pf}(M) = \frac{1}{2^n n!}\sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma \cdot M_{\sigma(1),\sigma(2)} \cdots M_{\sigma(2n-1),\sigma(2n)}.$$
- If a skew-symmetric M has odd dimensions, set $\mathrm{Pf}(M) = 0$. Then $\mathrm{Pf}(M)^2 = \det(M)$ for all skew-symmetric matrices.

$$\mathrm{Pf}\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix} = b$$
$$\mathrm{Pf}\begin{pmatrix}0 & b & c & d\\ -b & 0 & g & h\\ -c & -g & 0 & l\\ -d & -h & -l & 0\end{pmatrix} = bl - ch + dg.$$

Consequences

Topological form

- Immediate algebraic properties i.e. convergence
- Explicit, easily computable formula for all ℓ
- Quadratic relations coming from Stokes' relations:
$$\delta I_G + \frac{1}{2}[I_G, \mathfrak{m}] = 0, \quad I_G = \langle G, \mathfrak{m}\rangle$$
Equivalent to Maurer-Cartan equation for $\mathfrak{m}$, a sum over even-looped multiedges (dipoles).

Pfaffian form

- Interpretation of ϕ_G as parametric integrand corresponding to single topological dimension of integrals computing violations of BRST-closedness.
- $\phi_G \wedge \phi_G = 0$ gives simpler proof and generalizes Kontsevich's formality result
- Position space representation of $I_G = \int_{\sigma_G}\phi_G$

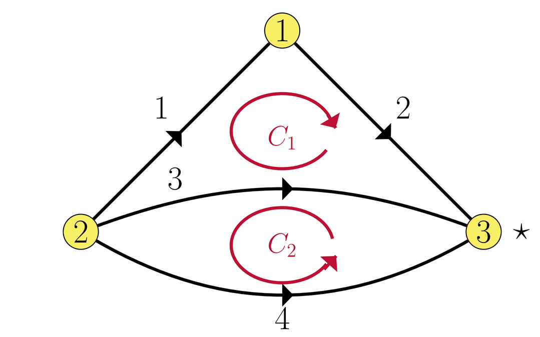
Proof ingredients: Graph matrices

Let E_G be set of edges, V_G set of vertices. Leave out one vertex $v_\star$ (physics interpretation: Fix at the origin) Loop number $\ell = |E_G| - (|V_G| + 1)$. Assign one Schwinger parameter a_e to each edge e.

$$\mathcal{D} = \begin{pmatrix}a_1 & 0 & 0 & 0\\ 0 & a_2 & 0 & 0\\ 0 & 0 & a_3 & 0\\ 0 & 0 & 0 & a_4\end{pmatrix}, \qquad \mathbb{I} = \begin{pmatrix}1 & -1\\ -1 & 0\\ 0 & -1\\ 0 & -1\end{pmatrix}$$
Schwinger parameters — edge-vertex incidence

$$\mathcal{C} = \begin{pmatrix}1 & 0\\ 1 & 0\\ -1 & -1\\ 0 & 1\end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix}1 & 0\\ 0 & 0\\ -1 & 0\\ 0 & -1\end{pmatrix}$$
edge-cycle incidence — paths v to $v_\star$

Dunce's cap G is a graph on 3 vertices and 4 edges, with $\ell = 2$ loops.

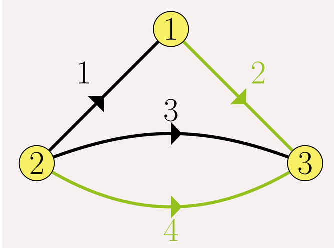


- Laplacians: $\mathcal{L} = \mathbb{I}^\intercal\mathcal{D}^{-1}\mathbb{I} = \begin{pmatrix}\frac{1}{a_1} + \frac{1}{a_2} & -\frac{1}{a_1}\\ -\frac{1}{a_1} & \frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_4}\end{pmatrix}$, $\quad \Lambda = \mathcal{C}^\intercal\mathcal{D}\mathcal{C} = \begin{pmatrix}a_1 + a_2 + a_3 & a_3\\ a_3 & a_3 + a_4\end{pmatrix}$.
- Symanzik polynomial: $\psi_G = \det\Lambda = \det\mathcal{L} \cdot \prod_{e \in E_G} a_e = a_3a_4 + a_1(a_3 + a_4) + a_2(a_3 + a_4)$.
- Matrix tree theorem: The monomials of ψ are the complements of spanning trees, $\psi = \sum_T \prod_{e \notin T} a_e$.

Expanded vertex Laplacian:
$$\mathbb{M} := \begin{pmatrix}\mathcal{D} & \mathbb{I}\\ -\mathbb{I}^T & 0\end{pmatrix} = \begin{pmatrix}a_1 & 0 & 0 & 0 & 1 & -1\\ 0 & a_2 & 0 & 0 & 1 & 0\\ 0 & 0 & a_3 & 0 & 0 & 1\\ 0 & 0 & 0 & a_4 & 0 & 1\\ -1 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & -1 & 0 & 0\end{pmatrix}.$$
In $\mathbb{M}$, the first 4 rows and columns refer to edges, the last 2 rows and columns refer to vertices v_1, v_2.

Dodgson Polynomials: Minors of $\mathbb{M}$. Example:
$$\psi^{v_1,v_1} = \det\begin{pmatrix}a_1 & 0 & 0 & 0 & -1\\ 0 & a_2 & 0 & 0 & 0\\ 0 & 0 & a_3 & 0 & 1\\ 0 & 0 & 0 & a_4 & 1\\ 1 & 0 & -1 & -1 & 0\end{pmatrix} = a_2\left(a_1a_3 + a_1a_4 + a_3a_4\right),$$
$$\psi^{v_1,v_2} = -a_2a_3a_4 = \psi^{v_2,v_1}, \qquad \psi^{v_2,v_2} = (a_1 + a_2)a_3a_4.$$
They satisfy numerous identities.

Example: Topological/Pfaffian form for the Dunce's cap



G has five spanning trees T. For example, consider $T = \{2, 4\}$.

Then $E \setminus T = \{f_1, f_2\} = \{1, 3\}$ and $\mathbb{I}[T] = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ and $\psi^{1,3} = -a_4$.

Contribution of T: $\dfrac{(+1)}{16\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}} \cdot (-2a_4)\,\mathrm{d}a_1 \wedge \mathrm{d}a_3$.

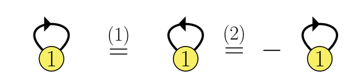
Sum of all trees: $$\alpha_G = \frac{-a_4(\mathrm{d}a_1 \wedge \mathrm{d}a_3 + \mathrm{d}a_2 \wedge \mathrm{d}a_3) + a_3(\mathrm{d}a_1 \wedge \mathrm{d}a_4 + \mathrm{d}a_2 \wedge \mathrm{d}a_4) - (a_1 + a_2)\,\mathrm{d}a_3 \wedge \mathrm{d}a_4}{8\pi(a_1a_3 + a_2a_3 + a_1a_4 + a_2a_4 + a_3a_4)^{3/2}}.$$

$$\Lambda^{-1} = \frac{1}{\psi_G}\begin{pmatrix}a_3 + a_4 & -a_3\\ -a_3 & a_1 + a_2 + a_3\end{pmatrix} \quad \text{and} \quad \mathrm{d}\Lambda = \begin{pmatrix}\mathrm{d}a_1 + \mathrm{d}a_2 + \mathrm{d}a_3 & \mathrm{d}a_3\\ \mathrm{d}a_3 & \mathrm{d}a_3 + \mathrm{d}a_4\end{pmatrix} \quad \text{gives} \quad \phi_G = 4\alpha_G.$$

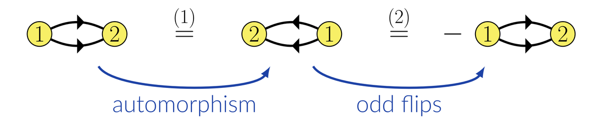
Commutative graph complexes

The **odd graph complex** GC_3 is a quotient of a $\mathbb{Q}$-vector space spanned by *oriented* graphs (G, o) [Kon93]
$$\mathrm{GC}_3 := \bigoplus_{(G,o)} \mathbb{Q}(G,o)/\sim, \qquad \text{where the orientation } o \in \det\mathbb{Z}^{V_G} \otimes \bigotimes_{e \in E_G} \det\mathbb{Z}^{H(e)} \cong \mathbb{Z}$$
and G is connected with vertex valencies ≥ 3. An orientation o is given by (vertex ordering + edge directions), or equivalently (cycle basis + edge ordering) [CV03]. GC_3 is bigraded by loop number ℓ and $k := \deg(G) = |E_G| - 3\ell$.

- The relations are: modulo isomorphisms $f\colon G \cong G'$ by $(G,o) \overset{(1)}{\sim} (G', f_*(o))$ and modulo orientation flips $(G,o) \overset{(2)}{\sim} -(G,-o)$. This implies that all graphs with tadpoles (or other *odd* automorphisms) vanish:

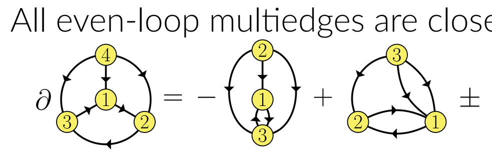


- Multi edges do not vanish automatically, but graphs which are *only* multi edges with even number of edges (=odd number of loops) vanish:



- Let G/γ denote contraction of subgraph $\gamma \subset G$ to a vertex. Define the boundary operator
$$\partial(G,o) = \sum_{e \in E_G}(G,o)/e.$$
Example: All even-loop multiedges are closed.

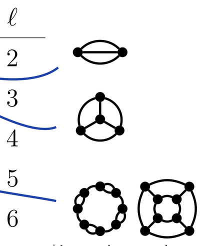
Example:



- *Graph homology* is
$$H_\bullet(\mathrm{GC}_3) = \frac{\ker\partial}{\mathrm{im}\,\partial} = \bigoplus_{\ell,k}\mathrm{gr}_\ell H_k(\mathrm{GC}_3)$$
- Homologies are known up to $\ell \approx 11$ [Wil25]. One finds only few classes, but for $\ell \to \infty$, their dimension grows super-exponentially [BZ24].
- H^{-3} related to "algebra of 3-graphs" [DKC98] and thus Vassiliev invariants in knot theory [Vog11].

Homologies of GC_3:

$\cdots$	H_{-9}	H_{-8}	H_{-7}	H_{-6}	H_{-5}	H_{-4}	H_{-3}	ℓ
						0	1	2
				0	0	0	1	3
		0	0	0	0	0	1	4
	0	0	0	0	0	0	2	5
	0	0	0	1	0	0	2	6
	0	0	0	1	0	0	3	7
	0	0	0	2	0	0	4	8
	0	0	0	3	0	0	5	9



The **even graph complex** GC_2 is defined similarly but with orientation $o \in \deg\mathbb{Z}^{E_G}$ given by an edge ordering and with degree $k = |E_G| - 2\ell$. Now multiedges vanish and tadpoles do not.

Orientation integrals on the odd graph complex: The Pfaffian form

- Connected graph with loop number ℓ, and differential wrt Schwinger parameters
$$\Lambda := \mathcal{C}^\intercal\mathcal{D}\mathcal{C}, \qquad \mathrm{d}\Lambda = \mathrm{d}\left(\mathcal{C}^\intercal\mathcal{D}\mathcal{C}\right) = \mathcal{C}^\intercal\,\mathrm{d}\mathcal{D}\mathcal{C}.$$
Λ is a symmetric $\ell \times \ell$ matrix (and positive-definite) and $\mathrm{d}\Lambda \cdot \Lambda^{-1} \cdot \mathrm{d}\Lambda$ is skew-symmetric when ℓ is even.
- The **Pfaffian form** ϕ_G [BHP24] and the *primitive* **canonical forms** β_G^{4k+1} [Bro21] are defined as
$$\phi_G =: \frac{1}{(-2\pi)^{\ell/2}}\frac{\mathrm{Pf}\left(\mathrm{d}\Lambda \cdot \Lambda^{-1} \cdot \mathrm{d}\Lambda\right)}{\sqrt{\det\Lambda}} \quad \text{and} \quad \beta_G^{4k+1} := \mathrm{tr}\left(\left(\Lambda^{-1}\mathrm{d}\Lambda\right)^{4k+1}\right), \quad \text{for } k \geq 1.$$
Note $\beta_X^n = 0$ for symmetric matrices X if $n \neq 4k + 1$.
- Change of cycle basis $\mathcal{C}' = \mathcal{C}P$ with constant matrix $P \in \mathrm{GL}_\ell(\mathbb{Z})$:
$$\begin{aligned}\phi_{\Lambda'} &= \phi_\Lambda \cdot \det P = \pm\phi_\Lambda\\ \beta_{\Lambda'}^{4k+1} &= \beta_\Lambda^{4k+1}\end{aligned} \quad \Leftarrow \quad \begin{aligned}&\mathrm{d}\Lambda'\Lambda'^{-1}\mathrm{d}\Lambda' = P^\intercal\left(\mathrm{d}\Lambda\Lambda^{-1}\mathrm{d}\Lambda\right)P,\ \Lambda'^{-1}\mathrm{d}\Lambda' = P^{-1}(\Lambda^{-1}\mathrm{d}\Lambda)P\\ &\text{known: } \mathrm{Pf}(A^\intercal BA) = \det A\,\mathrm{Pf}(B), \quad \text{trace is cyclic}\end{aligned}$$
Any wedge product of these forms, **orientation forms** $\phi \wedge \omega$, changes sign by $\det P$ under changes of basis.
- Closed forms: $\mathrm{d}\phi = 0$ and $\mathrm{d}\beta^{4k+1} = 0$, and generate a Hopf algebra of forms where β are primitive.
- Integral over simplex $\sigma_G = \{[a_1 : \ldots : a_{|E_G|}], a_e > 0\} \in \mathbb{P}(\mathbb{R}_+^{|E_G|})$ is always finite and satisfies *Stokes' relation*
$$I_G(\omega) = \int_{\sigma_G}\phi_G \wedge \omega_G, \quad \delta I(\omega) + [I(\omega), \mathfrak{m}] + \frac{1}{2}\sum_{(\omega)}(-1)^{|\omega'|}\left[I(\omega''), I(\omega')\right] = 0$$
where $\mathfrak{m}$ is a sum over even-looped multiedges (dipoles), weighted by automorphism factors.
- These are well-defined on GC_3; under cocycle conditions, is an *integration pairing* that computes homology!
- Generalized Feynman integrals: $\displaystyle\int_{\sigma_G}\phi_G \wedge \omega_G = \int_{\sigma_G}\frac{Q(a_e)}{\psi^{\ell+1/2}}\,\Omega_{|E_G|}$ where $Q(a_e)$ is a polynomial.

Why are integrals detecting homology?

- Let G be some (linear combination of) graphs such that $\partial G = 0$, i.e. checked by explicit computation. Hard part: Does there $\exists F$ such that $\partial F = G$?
- Stokes' theorem: Let $\int_F = \int_{\sigma_F}$ and $\mathrm{d}\omega = 0$,
$$0 = \int_F \mathrm{d}\omega = \int_{\partial F}\omega = \int_G \omega, \qquad \text{if } \partial F = G.$$
- Thus if $\int_G \omega \neq 0$ one knows that $G \neq \partial F$.
That is, G is not exact, and since $\partial G = 0$, this G defines a homology class in the even/odd graph complex (depending on ω).

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